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SOLVING FUZZY DIFFERENTIAL EQUATIONS USING RUNGE-KUTTA FOURTH ORDER GILL METHOD

D.Paul Dhayabaran*, J.Christy kingston
* Associate Professor & Principal, PG and Research Department of Mathematics,
Bishop Heber College (Autonomous), Tiruchirappalli -620 017,India
Assistant Professor, PG and Research Department of Mathematics,
Bishop Heber College (Autonomous), Tiruchirappalli -620 017,India

ABSTRACT
In this paper, a numerical solution for the first order fuzzy differential equations by using fourth order Runge-kutta Gill method is considered.The applicability and accuracy of the proposed method has been demonstrated by an example and the convergence of the method has been studied with a triangular fuzzy number.

KEYWORDS: Fuzzy Differential Equations, Runge-kutta fourth order Gill Method,Triangular Fuzzy number.

INTRODUCTION
Fuzzy differential equations are a natural way to model dynamical systems under uncertainty.First order linear fuzzy differential equation is one of the simplest fuzzy differential equations,which appear in many applications.The concept of fuzzy derivative was first introduced by S.L.Chang and L.A.Zadeh in[6].D.Dubois and Prade [7] discussed differentiation with fuzzy features.M.L.puri , D.A.Ralescu[24] and R.Goetschel, W.Voxman[10],these authors contributed towards the differential of fuzzy functions.The fuzzy differential equation and initial value problems were extensively studied by O.Kaleva[15,16] and by S.Seikkala[25].Recently many research papers are focused on numerical solution of fuzzy initial value problems .Numerical Solution of fuzzy differential equations has been introduced by M.Ma, M.Friedman, A.Kandel [19] through Euler method and S.Abbasbandy,T.Allahviranloo [1] by Taylor's method.Runge –Kutta methods have also been studied by authors [2,22].V.Nirmala,N.Saveetha,S.Chenthurpandiyan discussed on numerical solution of fuzzy differential equations by Runge-Kutta method with higher order derivative approximations[21].R.Gethsi sharmila and E.C.Henry Amirtharaj discussed on numerical solutions of first order fuzzy initial value problems by non-linear trapezoidal formulae based on variety of means[13].Fourth order Runge-kutta Gill method was proposed by Oliveria.S.C [17],and also it was studied by R.Ponalagusamy, S.Senthilkumar [23].Following by the introduction this paper is organised as follows:In section 2, some basic results of fuzzy numbers and definitions of fuzzy derivative are given.In section 3, the fuzzy initial value problem is being discussed.Section 4 describes the general structure of the fourth order Runge-kutta Gill method.In section 5,the fourth order Runge-kutta Gill method was proposed for solving fuzzy initial value problem and the numerical examples are provided to illustrate the validity and applicability of the new method, followed by the conclusion given in the last section.

PRELIMINARIES
Definition:(FUZZY NUMBER)

An arbitrary fuzzy number is represented by an ordered pair of functions $(\underline{u}(r),\overline{u}(r))$ for all $r\in[0,1]$ which satisfy the following conditions.

i) $\underline{u}(r)$ is a bounded left continuous non-decreasing function over $[0,1]$ with respect to any r.

ii) $\overline{u}(r)$ is a bounded right continuous non-decreasing function over $[0,1]$ with respect to any r.

iii) $(\underline{u}(r) \le \overline{u}(r))$ for all $r \in [0,1]$ then the r-level set is $[u]_r = \{x \setminus u(x) \ge r \};0 \le r \le 1$

Clearly, $[u]_0 = \{x \setminus u(x) \ge 0\}$ *is compact,* which is a closed bounded interval and we denote by $[u]_r = (\underline{u}(r), \overline{u}(r))$

Definition: (TRIANGULAR FUZZY NUMBER)

A triangular fuzzy number u is a fuzzy set in E that is characterized by an ordered triple $(u_l, u_c, u_r) \in R^3$ with $u_l \le u_c \le u_r$ such that $[u]_0 = [u_l; u_r]$ and $[u]_l = \{u_c\}$.

The membership function of the triangular fuzzy number u is given by

$$u(x) = \begin{cases} \dfrac{x-u_l}{u_c-u_l} & ; \quad u_l \le x \le u_c \\ 1 & ; \quad x = u_c \\ \dfrac{u_r - x}{u_r - u_c} & ; \quad u_c \le x \le u_r \end{cases}$$

. we have :

(1) $u > 0 \ \text{if} \ u_l > 0;$

(2) $u \ge 0 \ \text{if} \ u_l \ge 0;$

(3) $u < 0 \ \text{if} \ u_c < 0$;and

(4) $u \le 0 \ \text{if} \ u_c \le 0$

Definition: (α - Level Set)

Let I be the real interval. A mapping $y: I \to E$ is called a fuzzy process and its α - *level Set* is denoted by $[y(t)]_\alpha = [\underline{y}(t;\alpha), \overline{y}(t;\alpha)\ ]$, $t \in I$, $0 < \propto < 1$

Definition: (Seikkala Derivative)

The Seikkala derivative $y'(t)$ of a fuzzy process is defined by $[y'(t)]_\propto = [\underline{y}'(t;\alpha), \overline{y}'(t;\alpha)\ ]$ $t \in I$, $0 < \propto \le 1$ provided that this equation defines a fuzzy number, as in [25]

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Lemma:

If the sequence of non-negative number $\{W_n\}_{n=0}^{m}$ satisfy $|Wn+1| \leq A|Wn| + B$ $, 0 \leq n \leq N-1$ for the given positive constants *A* and *B*, then $|Wn| \leq A^n |W_0| + B\dfrac{A^n - 1}{A-1}$ $, 0 \leq n \leq N$

Lemma:

If the sequence of non-negative numbers $\{W_n\}_{n=0}^{m}, \{V_n\}_{n=0}^{N}$ satisfy $|W_{n+1}| \leq |W_n| + A\, max\{|W_n|, |V_n|\} + B$,

$|V_{n+1}| \leq |V_n| + A\, max\{|W_n|, |V_n|\} + B$ for the given positive constants *A* and *B*, then $U_n = |W_n| + |V_n|$, $0 \leq n \leq N$

we have, $U_n \leq \overline{A}^n U_0 + B\dfrac{\overline{A}^n - 1}{A-1}$ $0 \leq n \leq N$ where $\overline{\mathrm{A}} = 1 + 2\mathrm{A}$ and $\overline{B} = 2B$.

Lemma

Let $F(t,u,v)$ and $G(t,u,v)$ belong to $C'(R_F)$ and the partial derivatives of *F* and *G* be bounded over R_F ,then for arbitrarily fixed *r*, $0 \leq r \leq 1$, $D(y(t_{n+1}), y^0(t_{n+1})) \leq h^2 L(1+2C)$ where *L* is a bound of partial derivatives of *F* and *G*, and $C = Max\left\{\left|G\left[t_N, \underline{y}(t_N;r), \overline{y}(t_{N-1};r)\right]\right|, r \in [0,1]\right\} < \infty$

Theorem

Let $F(t,u,v)$ and $G(t,u,v)$ belong to $C'(R_F)$ and the partial derivatives of *F* and *G* be bounded over R_F ,then for arbitrarily fixed *r*, $0 \leq r \leq 1$, the numerical solutions of $\underline{y}(t_{n+1};r)$ and $\overline{y}(t_{n+1};r)$ converge to the exact solutions $\underline{Y}(t_{n+1};r)$ and $\overline{Y}(t_{n+1};r)$ uniformly in *t*.

Theorem

Let $F(t,u,v)$ and $G(t,u,v)$ belong to $C'(R_F)$ and the partial derivatives of *F* and *G* be bounded over R_F and $2Lh < 1$. Then for arbitrarily fixed $0 \leq r \leq 1$, the iterative numerical solutions of $\underline{y}^{(j)}(t_n;r)$ and $\overline{y}^{(j)}(t_n;r)$ converge to the numerical solutions $\underline{y}(t_n;r)$ and $\overline{y}(t_n;r)$ in $t_0 \leq t_n \leq t_N$, when $j \to \infty$.

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FUZZY INITIAL VALUE PROBLEM

A first-order fuzzy initial value differential equation is given by

$$\begin{cases} y'(t) = f(t, y(t)), \quad t \in [t_0, T] \\ y(t_0) = y_0 \end{cases} \tag{3.1}$$

where y is a fuzzy function of t, $f(t, y)$ is a fuzzy function of the crisp variable t and the fuzzy variable y, y' is the fuzzy derivative of y and $y(t_0) = y_0$ is a triangular or a triangular shaped fuzzy number.

We denote the fuzzy function y by $y = [\underline{y}, \overline{y}]$.It means that the r-level set of $y(t)$ for $t \in [t_0, T]$ is

$$[y(t)]_r = [\underline{y}(t;r), \overline{y}(t;r)],$$

$$[y(t_0)]_r = [\underline{y}(t_0;\ r), \overline{y}(t_0;\ r)],\ r \in (0,1],$$

we write $f(t,\ y) = [\underline{f}(t, y), \overline{f}(t, y)]$ and

$$\underline{f}\ (t, y) = F[t, \underline{y}, \overline{y}], \quad \overline{f}(t, y) = G[t, \underline{y}, \overline{y}]$$

because of $y' = f(t, y)$ we have

$$\underline{f}\ (t, y(t); r) = F[t,\ \underline{y}(t;r), \overline{y}(t;r)] \tag{3.2}$$

$$\overline{f}\ (t, y(t); r) = G[t,\ \underline{y}(t;r), \overline{y}(t;r)] \tag{3.3}$$

by using the extension principle, we have the membership function

$$f(t, y(t))(s) = sup\{y(t)(\tau) \setminus s = f(t, \tau)\},\ s \in R \tag{3.4}$$

so the fuzzy number $f(t, y(t))$ follows that

$$[f(t, y(t))]_r = [\underline{f}(t, y(t); r), \overline{f}(t, y(t); r)], \quad r \in (0,1] \tag{3.5}$$

where $\underline{f}(t, y(t); r) = \ min\{f(t,u) \,|\, u[y(t)]_r\}$ (3.6)

$$\overline{f}(t, y(t); r) = \ max\{f(t,u) \,|\, u[y(t)]r\ \}. \tag{3.7}$$

Definition 3.1 A function $f: R \to R_F$ is said to be fuzzy continuous function, if for an arbitrary fixed $t_0 \in R$ and $\varepsilon > 0$, $\delta > 0$ such that $|t - to| < \delta \Rightarrow D[f(t), f(t_0)] < \varepsilon$ exists.

The fuzzy function considered are continuous in metric D and the continuity of $f(t, y(t); r)$ guarantees the existence of the definition of $f(t, y(t); r)$ for $t \in [t_0, T]$ and $r \in [0,1]$ [10]. Therefore, the functions *G* and *F* can be definite too.

FOURTH ORDER RUNGE-KUTTA GILL METHOD

Fourth order Runge-kutta Gill method was proposed for approximating the solution of the initial value problem $y'(t) = f(t, y(t)),\ y(t_0) = y_0$. The basis of all Runge-Kutta methods is to express the difference between the value of y at t_{n+1} and t_n as

$$y_{n+1} - y_n = \sum_{i=0}^{m} w_i k_i \tag{4.1}$$

where w_i's are constant for all i and

$$k_i = hf(t_n + a_i h, y_n + \sum_{j=1}^{i-1} c_{ij} k_j) \tag{4.2}$$

Increase of the order of accuracy of the Runge-Kutta methods have been accomplished by increasing the number of taylor's series terms used and thus the number of functional evaluations required[5].The method proposed by Goeken.D and Johnson.O[9] introduces new terms involving higher order derivatives of 'f' in the Runge-Kutta $k_i$terms(i > 0) to obtain a higher order of accuracy without a corresponding increase in evaluations of 'f ', but with the addition of evaluations of f'.

The fourth order Runge-kutta Gill method for step n+1 which was proposed by Oliveria.S.C [17] is given by

$$y(t_{n+1}) = y(t_n) + \frac{1}{6}\left[k_1 + (2 - \sqrt{2})k_2 + (2 + \sqrt{2})k_3 + k_4\right] \tag{4.3}$$

Where

$$k_1 = hf(t_n, y(t_n)) \tag{4.4}$$

$$k_2 = hf(t_n + a_1 h, y(t_n) + a_1 k_1) \tag{4.5}$$

$$k_3 = hf(t_n + (a_2 + a_3)h, y(t_n) + a_2 k_1 + a_3 k_2) \tag{4.6}$$

$$k_4 = hf(t_n + (a_4 + a_5)h, y(t_n) + a_4 k_2 + a_5 k_3) \tag{4.7}$$

The parameters a_1, a_2, a_3, a_4, a_5 are chosen to make y_{n+1} closer to y(t_{n+1}) .The value of parameters are

$$a_1 = \frac{1}{2},\ a_2 = \frac{1}{\sqrt{2}} - \frac{1}{2},\ a_3 = 1 - \frac{1}{\sqrt{2}},\ a_4 = \frac{-1}{\sqrt{2}},\ a_5 = 1 + \frac{1}{\sqrt{2}}$$

FOURTH ORDER RUNGE-KUTTA GILL METHOD FOR SOLVING FUZZY DIFFERENTIAL EQUATIONS

Let the exact solution $[Y(t)]_r = [\underline{Y}(t;r), \overline{Y}(t;r)]$,is approximated by some $[y(t)]_r = [\underline{y}(t;r), \overline{y}(t;r)]$, the grid points at which the solutions is calculated are $h = \frac{T - t_0}{N}$, $t_i = t_0 + ih; 0 \le i \le N$

From 4.3to 4.7 we define

$$\underline{y}(t_{n+1}, r) - \underline{y}(t_n, r) = \frac{1}{6}\begin{bmatrix} \underline{k_1}(t_n, y(t_n, r)) + (2 - \sqrt{2})\underline{k_2}(t_n, y(t_n, r)) \\ +(2 + \sqrt{2})\underline{k_3}(t_n, y(t_n, r)) + \underline{k_4}(t_n, y(t_n, r)) \end{bmatrix} \tag{5.1}$$

where

$$k_1 = hF[t_n, \underline{y}(t_n, r), \overline{y}(t_n, r)] \tag{5.2}$$

$$k_2 = hF[t_n + \frac{h}{2}, \underline{y}(t_n, r) + \frac{1}{2}\underline{k_1}(t_n, y(t_n, r)), \overline{y}(t_n, r) + \frac{1}{2}\overline{k_1}(t_n, y(t_n, r))] \tag{5.3}$$

$$\begin{aligned} k_3 = hF[t_n + \frac{h}{2}, & \underline{y}(t_n, r) + \left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right)\underline{k_1}(t_n, y(t_n, r)) + \left(1 - \frac{1}{\sqrt{2}}\right)\underline{k_2}(t_n, y(t_n, r)), \\ & \overline{y}(t_n, r) + \left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right)\overline{k_1}(t_n, y(t_n, r)) + \left(1 - \frac{1}{\sqrt{2}}\right)\overline{k_2}(t_n, y(t_n, r))] \end{aligned} \tag{5.4}$$

$$\begin{aligned} k_4 = hF[t_n + h, & \underline{y}(t_n, r) - \frac{1}{\sqrt{2}}\underline{k_2}(t_n, y(t_n, r)) + \left(1 + \frac{1}{\sqrt{2}}\right)\underline{k_3}(t_n, y(t_n, r)), \\ & \overline{y}(t_n, r) - \frac{1}{\sqrt{2}}\overline{k_2}(t_n, y(t_n, r)) + \left(1 + \frac{1}{\sqrt{2}}\right)\overline{k_3}(t_n, y(t_n, r))] \end{aligned} \tag{5.5}$$

and

$$\overline{y}(t_{n+1}, r) - \overline{y}(t_n, r) = \frac{1}{6}\begin{bmatrix} \overline{k_1}(t_n, y(t_n, r)) + (2 - \sqrt{2})\overline{k_2}(t_n, y(t_n, r)) \\ +(2 + \sqrt{2})\overline{k_3}(t_n, y(t_n, r)) + \overline{k_4}(t_n, y(t_n, r)) \end{bmatrix} \tag{5.6}$$

where

$$k_1 = hG[t_n, \underline{y}(t_n, r), \overline{y}(t_n, r)] \tag{5.7}$$

$$k_2 = hG[t_n + \frac{h}{2}, \underline{y}(t_n, r) + \frac{1}{2}\underline{k_1}(t_n, y(t_n, r)), \overline{y}(t_n, r) + \frac{1}{2}\overline{k_1}(t_n, y(t_n, r))] \tag{5.8}$$

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$$k_3 = hG[t_n + \frac{h}{2},\ \underline{y}(t_n,r) + \left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right)\underline{k_1}(t_n, y(t_n,r)) + \left(1 - \frac{1}{\sqrt{2}}\right)\underline{k_2}(t_n, y(t_n,r)),$$
$$\overline{y}(t_n,r) + \left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right)\overline{k_1}(t_n, y(t_n,r)) + \left(1 - \frac{1}{\sqrt{2}}\right)\overline{k_2}(t_n, y(t_n,r))] \quad (5.9)$$

$$k_4 = hG[t_n + h,\ \underline{y}(t_n,r) - \frac{1}{\sqrt{2}}\underline{k_2}(t_n, y(t_n,r)) + \left(1 + \frac{1}{\sqrt{2}}\right)\underline{k_3}(t_n, y(t_n,r)),$$
$$\overline{y}(t_n,r) - \frac{1}{\sqrt{2}}\overline{k_2}(t_n, y(t_n,r)) + \left(1 + \frac{1}{\sqrt{2}}\right)\overline{k_3}(t_n, y(t_n,r))] \quad (5.10)$$

we define

$$F[t_n, y(t_n,r)] = \frac{1}{6}\begin{bmatrix} \underline{k_1}(t_n, y(t_n,r)) + (2-\sqrt{2})\underline{k_2}(t_n, y(t_n,r)) \\ +(2+\sqrt{2})\underline{k_3}(t_n, y(t_n,r)) + \underline{k_4}(t_n, y(t_n,r)) \end{bmatrix} \quad (5.11)$$

$$G[t_n, y(t_n,r)] = \frac{1}{6}\begin{bmatrix} \overline{k_1}(t_n, y(t_n,r)) + (2-\sqrt{2})\overline{k_2}(t_n, y(t_n,r)) \\ +(2+\sqrt{2})\overline{k_3}(t_n, y(t_n,r)) + \overline{k_4}(t_n, y(t_n,r)) \end{bmatrix}$$
(5.12)

Therefore we have

$$\underline{Y}(t_{n+1},r) = \underline{Y}(t_n,\ r) + F[t_n, Y(t_n,r)] \quad (5.13)$$

$$\overline{Y}(t_{n+1},r) = \overline{Y}(t_n,\ r) + G[t_n, Y(t_n,r)] \quad (5.14)$$

And

$$\underline{y}(t_{n+1},r) = \underline{y}(t_n,\ r) + F[t_n, Y(t_n,r)] \quad (5.16)$$

$$\overline{y}(t_{n+1},r) = \overline{y}(t_n,\ r) + G[t_n, Y(t_n,r)] \quad (5.17)$$

Clearly $\underline{y}(t;r)$ and $\overline{y}(t;r)$ converge to $\underline{Y}(t;r)$ *and* $\overline{Y}(t;r)$ whenever $h \to 0$.

NUMERICAL EXAMPLE

Consider the fuzzy initial value problem

$$\begin{cases} y'(t) = y(t), & t \geq 0 \\ y(0) = (0.75+0.25r;\ 1.5-0.5r) \end{cases} \quad (6.1)$$

The exact solution is given by

$$Y(t,r) = [(0.75+0.25r)e^{t}, (1.5-0.5r)e^{t}] \quad (6.2)$$

At t=1we get

$$Y(1,r) = [(0.75+0.25r)e, (1.5-0.5r)e],\ 0 \leq r \leq 1 \quad (6.3)$$

The values of exact and approximate solution with h= 0.1 is given in Table :1.The exact and approximate solutions obtained by the proposed method is plotted in Figure:1.The errors of exact and approximate solutions is plotted in Figure:2.

Table:1 ***Exact and Approximate Solutions***

r	Exact Solution (t=1) $\underline{Y}(t,r)$ $\overline{Y}(t,r)$	Approximate Solution (h=0.1) $\underline{y}(t,r)$ $\overline{y}(t,r)$	Error 1	Error 2
0	2.038711 , 4.077423	2.038710 , 4.077420	1.563243e-006	3.126486e-006
0.1	2.106668 , 3.941509	2.106667 , 3.941506	1.615351e-006	3.022270e-006
0.2	2.174625 , 3.805595	2.174624 , 3.805592	1.667459e-006	2.918053e-006
0.3	2.242583 , 3.669680	2.242581 , 3.669678	1.719567e-006	2.813837e-006
0.4	2.310540 , 3.533766	2.310538 , 3.533764	1.771675e-006	2.709621e-006
0.5	2.378497 , 3.397852	2.378495 , 3.397850	1.823783e-006	2.605405e-006
0.6	2.446454 , 3.261938	2.446452 , 3.261936	1.875891e-006	2.501189e-006
0.7	2.514411 , 3.126024	2.514409, , 3.126022	1.928000e-006	2.396972e-006
.0.8	2.582368 , 2.990110	2.582366, , 2.990108	1.980108e-006	2.292756e-006
0.9	2.650325 , 2.854196	2.650323 , 2.854194	2.032216e-006	2.188540e-006
1.0	2.718282 , 2.718282	2.718280 , 2.718280	2.084324e-006	2.084324e-006

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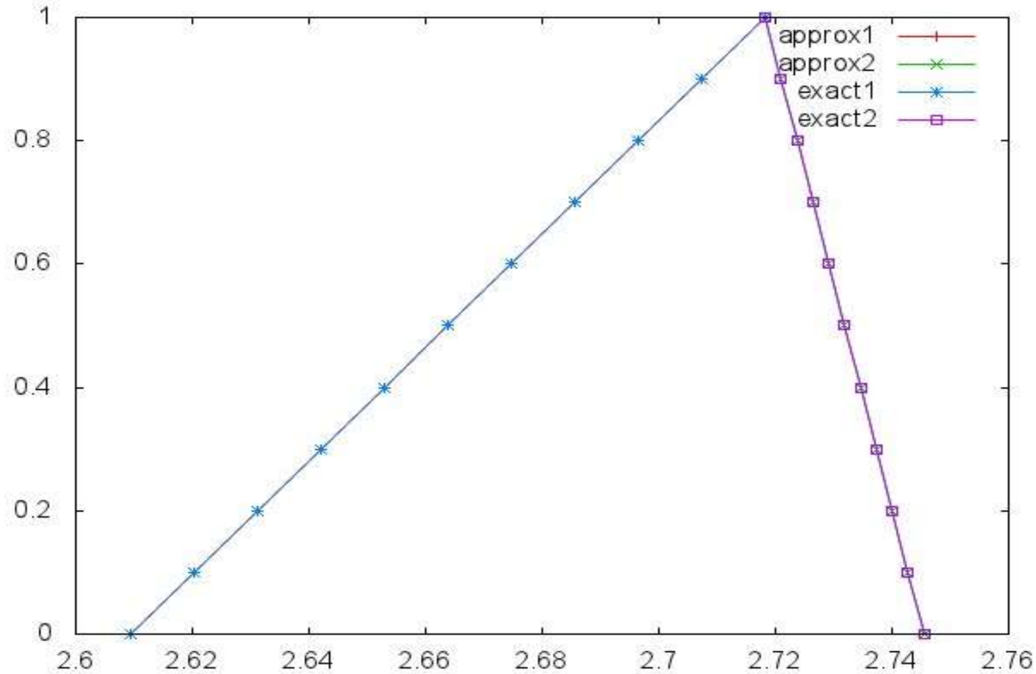
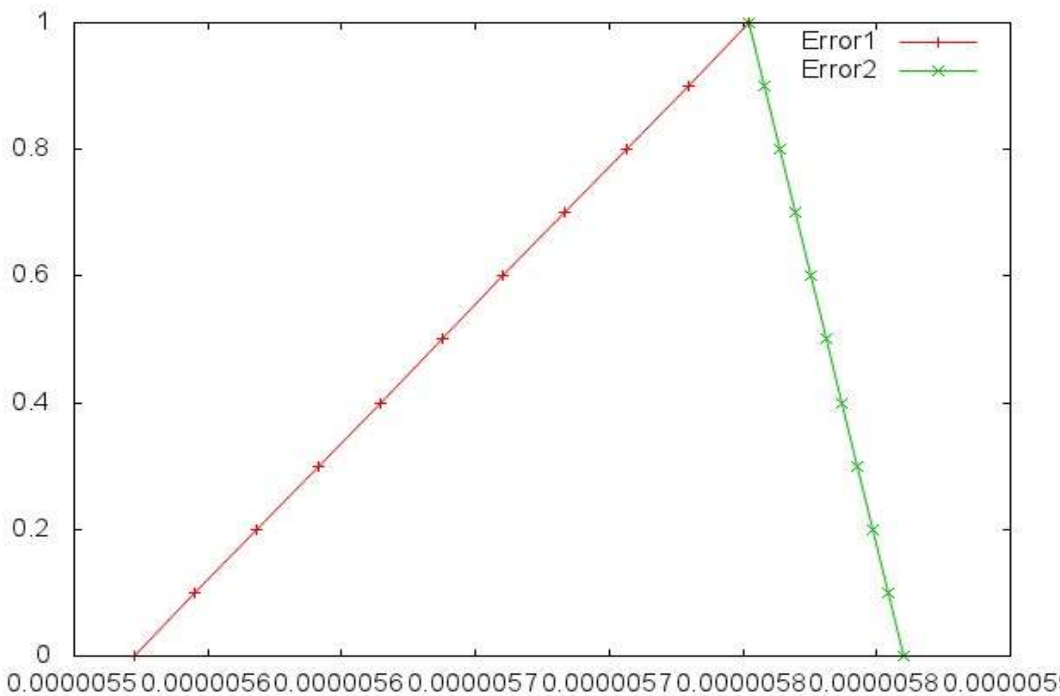
Figure:1



Exact and Approximate Solution

Figure:2



Erros of Exact and Approximate Solutions

CONCLUSION

In this paper the numerical solution of differential equations with fuzzy initial values were studied .The fourth order Runge-kutta gill method has been applied for finding the numerical solution of first order fuzzy differential equations using triangular fuzzy number.The efficiency and the accuracy of the proposed method have been illustrated by a suitable example.From the numerical example it has been observed that the discreet solutions by the proposed method almost coincide with the exact solutions.

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